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A note on topology and magnetic energy in incompressible perfectly conducting fluids

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In an incompressible perfectly conducting fluid the Navier–Stokes equations become

$$\frac{\partial \boldsymbol{v}}{\partial t}+\boldsymbol{v}.\nabla\boldsymbol{v}=-\nabla \mathrm{P}-\nu\Delta\boldsymbol{v}+\boldsymbol{j}\times\boldsymbol{H},$$

where $\boldsymbol{j}=\operatorname{curl}\boldsymbol{H}$, $\operatorname{div}\boldsymbol{H}=0$, $\operatorname{div}\boldsymbol{v}=0$, and $\mathrm{d}\boldsymbol{H}/\mathrm{d}t=\operatorname{curl}(\boldsymbol{v}\times\boldsymbol{H})$. The last equation follows from the Maxwell equation $\mathrm{d}\boldsymbol{H}/\mathrm{d}t=-\operatorname{curl}\boldsymbol{E}$ and the assumption of perfect conduction – that the electric field in the frame of the fluid vanishes, $\boldsymbol{E}'=\boldsymbol{E}+\boldsymbol{v}\times\boldsymbol{H}=0$. In geometric terms, it says that the system evolves so that the time derivative of $\boldsymbol{H}$ is equal to minus its spatial Lie derivative:

$$\frac{\mathrm{d}\boldsymbol{H}}{\mathrm{d}t}=-L_{\mathrm{v}}\boldsymbol{H}.$$

Thus $\boldsymbol{H}$ is equivariant with respect to the evolution (or 'frozen in the fluid') as long as the evolution follows these equations. Since the first equation tends to dissipate magnetic energy $E=\int\|\boldsymbol{H}\|^2$ the question naturally rises whether the topology of $\boldsymbol{H}$ determines lower bounds on E. We treat this question in the general context of a divergence-free vector field $\boldsymbol{H}$ on a closed Riemannian 3-manifold M. We obtain a result bounding E from below but make no assertion on the existence of extremals.

Arnol'd (1986) has defined a quadratic form for any 'null-homologous' $\boldsymbol{H}$,†

$$\mathrm{I}(\boldsymbol{H})=\int_{\mathrm{M}}\langle\operatorname{curl}^{-1}\boldsymbol{H},\boldsymbol{H}\rangle,$$

which is invariant under the group *S*Diff of volume-preserving diffeomorphisms. It follows that when $I(\boldsymbol{H})\neq 0$, E is bounded below on the *S*Diff orbit of $\boldsymbol{H}$. Arnol'd's invariant is a generalization of the homological linking number of two closed curves applied to the trajectories of $\boldsymbol{H}$. This has led Moffatt (1985) to conjecture that other 'higher-order' linking (not detectable homologically) also leads to positive lower bounds on E.

It is the purpose of this note to show that *any* non-trivial linking between circular packets of $\boldsymbol{H}$-integral curves implies a lower bound to E. An asymptotic version of this result – one not relying on closed orbits – would be most welcome and in keeping with the philosophy of Arnol'd's paper. To complete the context, Zel'dovich (see Arnol'd 1986, p. 331) has shown that if $\boldsymbol{H}$ is taken to be the killing field on S^3 generated by infinitesimal rotation about a one-dimensional axis (the circular orbits do not link!) then $\boldsymbol{H}$ may be deformed (by elements of *S*Diff S^3) to make the

† 'Null homologous' means the flux of $\boldsymbol{H}$ across any closed surface vanishes; this guarantees the existence of $\operatorname{curl}^{-1}\boldsymbol{H}$.

associated energy arbitrarily small, $E < \epsilon$. Possibly, configurations with no positive lower bound on energy are quite rare and amenable to classification.

By a 'link' is meant a smooth imbedding of n circles into a 3-manifold

$$\mathrm{L}\colon \bigcup_{i=1}^{n} \mathrm{S}^1 \to \mathrm{M}^3,\ n \geqslant 1.$$

The link is trivial if it bounds n smoothly and disjointly imbedded disks, $\bar{L}$,

$$\begin{array}{ccc} \bigcup_{i=1}^{n} \mathrm{S}^1 & \xrightarrow{\mathrm{L}} & \mathrm{M}^3 \\ \downarrow & \nearrow_{\bar{\mathrm{L}}} & \\ \bigcup_{i=1}^{n} \mathrm{D}^2 & & \end{array}$$

Otherwise the link is essential. We say a divergence-free vector field $\boldsymbol{H}$ on M is 'modelled on L' if there is a smooth imbedding of $\bigcup_{i=1}^{n} \mathrm{D}^2 \times \mathrm{S}^1$ on to a tubular neighbourhood of $\mathrm{L} \subset \mathrm{M}$ which carries the foliation by circles $\mathrm{pt} \times \mathrm{S}^1$ of $\bigcup_{i=1}^{n} \mathrm{D}^2 \times \mathrm{S}^1$ on to the integral curves of $\boldsymbol{H}$ near the link L.

THEOREM. *If* $\boldsymbol{H}$ *is a divergence-free field on a closed 3-manifold* M *which is modelled on an essential link* (*or knot*) L *then there is a positive lower bound to the energy* $E(f_* \boldsymbol{H})$ *over the orbit*, $f \in \mathcal{S}\mathrm{Diff}$, *of volume preserving diffeomorphisms of* M.

Note: Given any $\mathrm{L} \subset \mathrm{M}$ one may construct an $\boldsymbol{H}$ modelled on L. If T is a closed tubular neighbourhood of a link $\mathrm{L} \in \mathrm{M}$, it follows from Moser's (1965) result on the existence of volume-preserving diffeomorphisms (between diffeomorphic manifolds of equal volume) that T has a volume-preserving parameterization $\mathrm{p}\colon \bigcup_{i=1}^{n} (\mathrm{D}^2_{\mathrm{r}_i} \times \mathrm{S}^1) \to \mathrm{T}$. Let $\boldsymbol{J}$ be the vector field $\phi_i \partial/\partial\boldsymbol{\theta}$ where $\phi_i\colon \mathrm{D}_{\mathrm{r}_i} \to \mathrm{R}^+ \cup 0$ is a radial bump function on the disk which tapers smoothly to zero at radius r_i and $\partial/\partial\boldsymbol{\theta}$ is the unit tangent vector field to the second factor. The field $\boldsymbol{H}$ may be defined as $p_* \boldsymbol{J}$ on T and zero on $\mathrm{M} \backslash \mathrm{T}$.

Proof. We will prove the stronger result that the 1-norm $E_1(f_* \boldsymbol{H}) = \int_{\mathrm{M}} \|f_* \boldsymbol{H}\|$ has a positive lower bound.

Suppose $E_1 \to 0$. That is $\exists f_i \in \mathcal{S}\mathrm{Diff}$ such that $\int_M \|f_{i_*} \boldsymbol{H}\| \mathrm{d\,vol} \to 0$. Let $\mathrm{T} \subset \mathrm{M}$ be the invariant tubular neighbourhood of L. Let $q\colon \mathrm{X} = \bigcup_{i=1}^{n} (\mathrm{D}^2 \times \mathrm{S}^1)_i \to \mathrm{T}$ be a (not necessarily volume preserving) diffeomorphism which carries circles $\mathrm{pt} \times \mathrm{S}^1$ to orbits of $\boldsymbol{H}$. By compactness, $\exists\, c > 0$ such that

$$\frac{1}{c}\|\boldsymbol{H}\| \leqslant \left\| q_* \frac{\partial}{\partial \boldsymbol{\theta}} \right\| \leqslant c\|\boldsymbol{H}\| \quad \text{and} \quad \frac{1}{c} \mathrm{d\,vol}\,(t) \leqslant q^{-1^*} \mathrm{d\,vol}\,(x) \leqslant c \mathrm{d\,vol}\,(t)$$

at all points $t = q(x)$ of T. Thus

$$\int_X \left\| (f_i \circ q)_* \frac{\partial}{\partial \boldsymbol{\theta}} \right\| \mathrm{d\,vol}(x) \to 0.$$

Think of X as $Y \times S^1$ where $Y = \bigcup_{i=1}^n D_i^2$ with the natural measure. By Fubini's theorem: $\int_Y \text{length}\, f_i \circ q(y \times S^1) = \int_Y \text{length}\, \gamma_y^i \to 0$. So for any $\epsilon > 0$, $\exists_j$ such that for $i > j$ length $\gamma_y^i < \epsilon$ for $y \in Y^- \subset Y$ where Y^- has measure $(1-\epsilon)$.

Consider any component, ℓ_1, of L. For i large this component will be represented by many short (length $< \epsilon$) integral curves; let γ_1^i be one of them and let $_*$ be a base point on γ_1^i. If ϵ is sufficiently small, the geodesic ball of radius 4ϵ about $_*$, $B_{*,4\epsilon}$, cannot possibly contain all the short circles parallel to any component of L since these fill a volume bounded below independently of $i(\epsilon)$ whereas vol $(B_{*,4\epsilon}) \to 0$. (We may assume that 4ϵ is less than the injective radius of M so that B has the topology of a ball.) Thus the link f_iL is represented (for i sufficiently large) by n loops the last $n-1$ of which lie outside a 3ϵ-ball containing the first (since they are short and not contained in the 4ϵ-ball). By the same argument we may exclude a small (arbitrarily small if i is chosen sufficiently large) set of representing curves to obtain the additional condition that $\gamma_3^i, \ldots, \gamma_n^i$ lies outside the 3ϵ ball about a base point on γ_2^i. Proceeding in this way we find representative $\gamma_1^i, \ldots, \gamma_n^i$ each contained in a 3ϵ ball disjoint from the others. Hence f_iL for sufficiently large i, and therefore L, is completely split – there is a disjoint collection of smooth balls, the f_i-preimages of the balls of radius ϵ about base points on $\gamma_1^i, \ldots, \gamma_n^i$, each of which contains exactly one component and meets no others.

The condition 'completely split' does not in itself imply L is trivial since it may contain knotted components. Observe however if $\boldsymbol{H}$ is modelled on L it is modelled also on some link $2L$ of $2n$ components obtained from L by splitting each component into two (possibly twisted) parallel copies. (Simply define $D' \cup D'' \subset D^2$ to be any two disjointly imbedded subdisks of the unit disk and restrict q to $\cup_{i=1}^n (D' \cup D'') \times S^1$) to obtain a modelling on $2L$.) Our assumption that $E_1 \to 0$ implies that 2L is completely split. The following straightforward argument in 3-manifold topology shows that this implies L itself is trivial.

If a parallel γ' to a knot γ lies in a ball disjoint from γ then a radial homotopy in that ball together with a thin cylinder joining γ to γ' yields a 'Dehn disk' Δ – one whose singularities do not intersect its boundary. A fundamental theorem of 3-manifold topology, Dehn's Lemma [P] says that Δ can be replaced by an imbedded disk Δ' with $\partial \Delta' = \partial \Delta = \gamma$ and Δ' contained in an arbitrarily small neighbourhood of Δ – showing that the original knot bounds an imbedded disk and thus is trivial.

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